FINITE ELEMENT ANALYSIS OF ELASTO-PLASTIC BOUNDARY FOR SOME STRUCTURE PROBLEMS

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ABSTRACT

The finite element method (FEM) is used widely in analysis of elasto-plastic behaviours for structures. The analysis often involves a two-stage process: first, the internal force field acting on the structural material must be defined, and second, the response of the material to that force field must be determined. In other words, the analysis of behaviours of structural material is establishment relationships between stresses and strains in the structure in the plastic as well as elastic ranges. It furnishes more realistic estimates of load-carrying capacities of structures and provides a better understanding of the reaction of the structural elements to the forces induced in the material. An understanding of the role of the relevant mechanical variables that define the characteristic reaction of the material to the applied force is therefore essential to the engineer designing structures. A complete account of the theory and application of plasticity must deal with two equally important aspects: (1) the general technique used in the development of stress-strain relationships for inviscid elasto-plastic materials with work hardening as well as strain softening; and (2) the general numerical solution procedure for solving an elasto-plastic structural problem under the action of loads or displacements, each of which varies in a specified manner.

1. INTRODUCTION

It is generally regarded that the origin of plasticity, as a branch of mechanics of continua, dates back to a series of papers from 1864 to 1872 by Tresca on the extrusion of metals, in which he proposed the first yield condition. The actual formulation of the theory was done in 1870 by St. Venant, who introduced the basic constitutive relations for what today we would call rigid, perfectly plastic materials in plane stress. It remained for Levy later in 1870 to obtain the general equations in three dimensions. A generalization similar to the results of Levy was arrived at independently by von Mises in a landmark paper in 1913, accompanied by his well-known, pressure-insensitive yield criterion (J2-theory, or octahedral shear stress yield condition).

In 1924, Prandtl extended the St. Venant-Levy-von Mises equations for the plane continuum problem to include the elastic component of strain, and Reuss in 1930 carried out their extension to three dimensions. The appropriate flow rule associated with the Tresca yield condition, which contains singular regimes (i.e., corners or discontinuities in derivatives with respect to stress), was discussed by Reuss in 1932 and 1933 [1].

In 1958, Prager further extended this general framework to include thermal effects (non-isothermal plastic deformation), by allowing the yield surface to change its shape with temperature.

A very significant concept of work hardening, termed the material stability postulate, was proposed by Drucker in 1951 and amplified in his further papers. With this concept, the plastic stress-strain relations together with many related fundamental aspects of the subject may be treated in a unified manner [1].

2. FINITE ELEMENT ANALYSIS OF ELASTO-PLASTIC BOUNDARY

2.1 Finite element formulation for elements

Review on the Finite Element Method: Isoparametric formulation for three different element types.

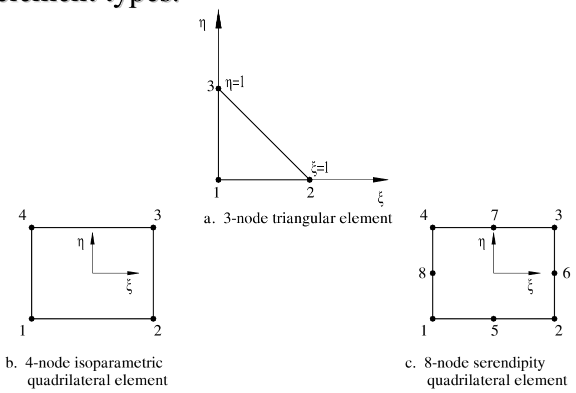

Fig. 1: 3-node triangular element and 4-node, 8-node quadrilateral element

The shape functions of the element :

+ 3-node triangular element:

$$N_1 = 1 - \xi - \eta;\ N_2 = \xi;\ N_3 = \eta \quad (1)$$

+ 4-node isoparametric quadrilateral element:

$$Ni = 0.25(1 + \xi\xi i)(1 + \eta\eta i) \quad (2)$$

+ 8-node serendipity quadrilateral element:

For corner nodes:

$$Ni = 0.25(1 + \xi\xi i)(1 + \eta\eta i)(\xi\xi i + \eta\eta i -1)$$
(3)

For corner nodes:

$$N_i = \frac{\xi_i^2}{2}(1 + \xi\xi_i)(1 - \eta^2) + \frac{\eta_i^2}{2}(1 + \eta\eta_i)(1 - \xi^2) \quad (4)$$

2.2 Formulation of the elasto-plastic matrix

2.2.1 3-D elasto-plastic stiffness matrix

The equation of the incremental stress-strain relation as follows [1]:

$$d\sigma ij = C^{ep}_{ijkl}\, d\varepsilon_{kl} = (C_{ijkl} + C^{p}_{ijkl})d\varepsilon_{kl} \quad (5)$$

in which the incremental stress and strain tensors $d\sigma_{ij}$, $d\varepsilon_{ij}$ are generally expressed in vector forms:

$$\{d\sigma\}^T = \{d\sigma_x\ d\sigma_y\ d\sigma_z\ d\tau_{yz}\ d\tau_{zx}\ d\tau_{xy}\ \} \quad (6)$$

$$\{d\varepsilon\}^T = \{d\varepsilon_x\ d\varepsilon_y\ d\varepsilon_z\ d\gamma_{yz}\ d\gamma_{zx}\ d\gamma_{xy}\ \} \quad (7)$$

and C_{ijkl} is the tensor of elastic modulus expressed in matrix form:

$$[C]=\begin{bmatrix} K+\frac{4}{3}G & K-\frac{2}{3}G & K-\frac{2}{3}G & 0 & 0 & 0 \\ K-\frac{2}{3}G & K+\frac{3}{4}G & K-\frac{2}{3}G & 0 & 0 & 0 \\ K-\frac{2}{3}G & K-\frac{2}{3}G & K+\frac{3}{4}G & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \quad (8)$$

where G and K are the shear and bulk moduli, respectively.

$$G = \frac{E}{2(1+\nu)} \quad (9)$$

$$K = \frac{E}{3(1-2\nu)}$$

(E is Young's modulus and ν Poisson's ratio)

C^{p}_{ijkl} is the plastic stiffness tensor.

$$[C^p]=-\frac{1}{H}\begin{bmatrix} \bar{s}_x^2 & & & & & \text{symmetric} \\ \bar{s}_y\bar{s}_x & \bar{s}_y^2 & & & & \\ \bar{s}_z\bar{s}_x & \bar{s}_z\bar{s}_y & \bar{s}_z^2 & & & \\ \bar{s}_{yz}\bar{s}_x & \bar{s}_{yz}\bar{s}_y & \bar{s}_{yz}\bar{s}_z & \bar{s}_{yz}^2 & & \\ \bar{s}_{zx}\bar{s}_x & \bar{s}_{zx}\bar{s}_y & \bar{s}_{zx}\bar{s}_z & \bar{s}_{zx}\bar{s}_{yx} & \bar{s}_{zx}^2 & \\ \bar{s}_{xy}\bar{s}_x & \bar{s}_{xy}\bar{s}_y & \bar{s}_{xy}\bar{s}_z & \bar{s}_{xy}\bar{s}_{yz} & \bar{s}_{xy}\bar{s}_{zx} & \bar{s}_{xy}^2 \end{bmatrix}$$

(10)

in which $\frac{1}{H} = \frac{36G^2}{h}$ (11)

2.2.2 Plane strain case

The condition of plane strain is:

$d\varepsilon_z = d\gamma_{yz} = d\gamma_{zx} = 0$ (12)

The elasto-plastic stiffness matrix is expressed in the form:

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$$[C^{ep}] = \begin{bmatrix} K + \frac{4}{3}G - \frac{\bar{s}_x^2}{H} & & \text{symmetric} \\ K - \frac{2}{3}G - \frac{\bar{s}_y \bar{s}_x}{H} & K + \frac{4}{3}G - \frac{\bar{s}_y^2}{H} & \\ -\frac{\bar{s}_{xy}\bar{s}_x}{H} & -\frac{\bar{s}_{xy}\bar{s}_y}{H} & G - \frac{\bar{s}_{xy}^2}{H} \end{bmatrix} \quad (13)$$

2.2.3 Plane stress case

The condition of plane stress is: $d\sigma_z = d\tau_{yz} = d\tau_{zx} = 0$; $d\varepsilon_z$ is nonzero

The elasto-plastic stiffness matrix is expressed in the form:

$$[C^{ep}] = \begin{bmatrix} \frac{E}{1-\nu^2} - \frac{\bar{s}_1^2}{\bar{s}} & & \text{sym.} \\ \frac{\nu E}{1-\nu^2} - \frac{\bar{s}_1\bar{s}_2}{\bar{s}} & \frac{E}{1-\nu^2} - \frac{\bar{s}_2^2}{\bar{s}} & \\ -\frac{\bar{s}_1\bar{s}_6}{\bar{s}} & -\frac{\bar{s}_2\bar{s}_6}{\bar{s}} & \frac{E}{2(1+\nu)} - \frac{\bar{s}_6^2}{\bar{s}} \end{bmatrix}$$

(14)

where

$$\bar{s}_1 = \frac{E}{1-\nu^2}(\bar{s}_x + \nu\bar{s}_y),$$

$$\bar{s}_2 = \frac{E}{1-\nu^2}(\nu\bar{s}_x + \bar{s}_y), \quad \bar{s}_6 = \frac{E}{2(1+\nu)}\bar{s}_{xy} \quad (15)$$

and $\bar{s} = \frac{4}{9}\bar{\sigma}_e H_p + \bar{s}_1\bar{s}_x + \bar{s}_2\bar{s}_y + 2\bar{s}_6\bar{s}_{xy}$ (16)

2.2.4 Axisymmetric case

The nonzero stress components in the axisymmetric case are σ_r, σ_z, σ_θ and τ_{rz} and corresponding strains are ε_r, ε_z, ε_θ and γ_{rz}. The matrix $[C^{ep}]$ is given by

$$[C^{ep}] = \begin{bmatrix} K + \frac{4}{3}G - \frac{\bar{s}_r^2}{H} & & & \text{symmetric} \\ K - \frac{2}{3}G - \frac{\bar{s}_z\bar{s}_r}{H} & K + \frac{4}{3}G - \frac{\bar{s}_z^2}{H} & K + \frac{4G}{3} - \frac{\bar{s}_z^2}{H} & \\ -\frac{\bar{s}_{rz}\bar{s}_r}{H} & -\frac{\bar{s}_{rz}\bar{s}_z}{H} & -\frac{\bar{s}_{rz}\bar{s}_\theta}{H} & G - \frac{\bar{s}_{rz}^2}{H} \end{bmatrix} \quad (17)$$

3. ELASTO-PLASTIC TIMOSHENKO BEAM ANALYSIS

3.1 Timoshenko beam theory

This theory allows for transverse shear deformation effects while Euler-Bernoulli beam theory takes no account of transverse shear deformation.

Stiffness matrices**:**

The governing equation.

$[K_f + K_s]\varphi - f = 0$ (18)

where, the submatrices of K_f and K_s and subvectors of f for element e.

Element stiffness matrix by using a 1-point Gauss-Legendre rule:

$$K_f^{(e)} = \left(\frac{EI}{l}\right)^{(e)} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad (19)$$

$K_s^{(e)}$ is evaluated exactly using a 2-point Gauss-Legendre rule:

$$K_s^{(e)} = \left(\frac{G\hat{A}}{l}\right)^{(e)} \begin{bmatrix} 1 & \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & \frac{l^2}{3} & -\frac{1}{2} & \frac{l^2}{6} \\ -1 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{l}{2} & \frac{l^2}{2} & -\frac{1}{2} & \frac{l^2}{3} \end{bmatrix}^{(e)} \quad (20)$$

$K_s^{(e)}$ is evaluated exactly using a 1-point Gauss-Legendre rule:

$$K_s^{(e)} = \left(\frac{G\hat{A}}{l}\right)^{(e)} \begin{bmatrix} 1 & \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & \frac{l^2}{4} & -\frac{1}{2} & \frac{l^2}{4} \\ -1 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{l}{2} & \frac{l^2}{4} & -\frac{1}{2} & \frac{l^2}{4} \end{bmatrix}^{(e)} \quad (21)$$

3.2 Elasto-plastic layered Timoshenko beams

3.2.1 Formulations in the layer approach

Bending moment M and shear force Q by using the mid-ordinate rule:

$$M = EI\left(-\frac{d\theta}{dx}\right) \text{and } Q = G\hat{A}\varepsilon_s \quad (22)$$

$$\text{where } EI = \sum_l E_l(b_l z_l^2 t_l)\ ;\quad G\hat{A} = \sum_l G_l b_l t_l \quad (23)$$

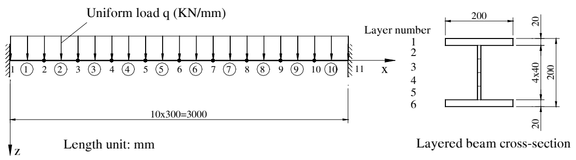
where
b_l is the layer breadth,
t_l is the layer thickness
z_l is the z-coordinate at the middle of the layer
E_l is Young's modulus of the layer material
G_l is the shear modulus of the layer material

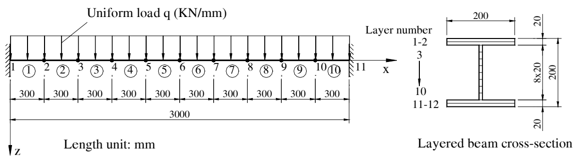
If the stress at the middle surface of a layer reaches the uniaxial yield stress of the layer material, the whole layer is considered to be plastic and E_l is replaced by: $E_l\left(1-\frac{E_l}{E_l+H'}\right)$

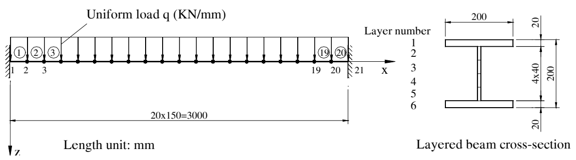
where H' is the uniaxial strain hardening parameter

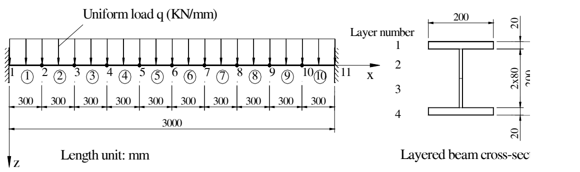
3.3 Beam problem

Finite element idealisation:









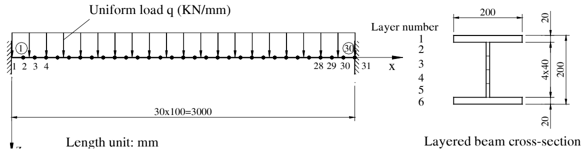


Fig. 2: Finite element idealisation of meshes M1, M2, M3, M4, M5

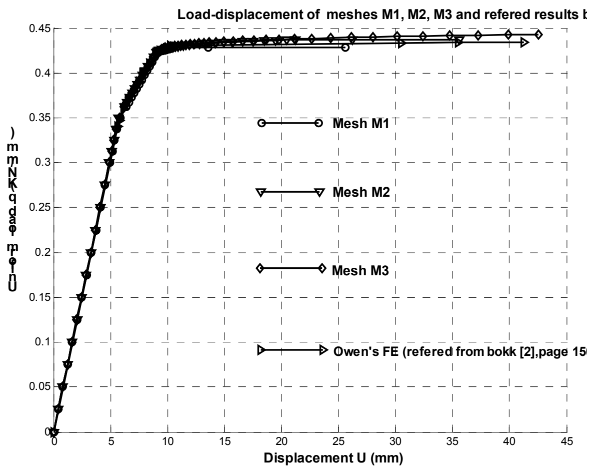


Fig. 3: Uniform load – displacement curves for meshes M_1, M_2, M_3 and Owen's FE

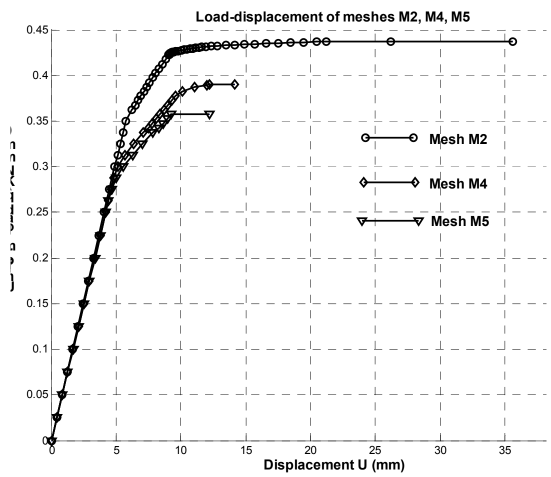


Fig. 4: Uniform load – displacement curves for meshes M2, M4 and M5

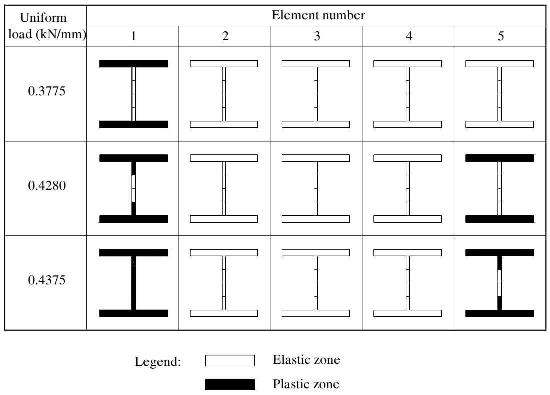


Fig. 5: Distribution of plastic layers of some sections at elements with various uniform load of mesh M2

Uniform load q (kN/mm)	Displacement U (mm)		Error (%)
	1-Gauss point (a)	2-Gauss point (b)	a-b
0.4290	10.70154906	10.63478666	0.623857
0.4320	12.31206323	12.13690619	1.422646
0.4325	12.83619912	12.57407891	2.042039
0.4365	19.45118787	18.37697980	5.522583
0.4370	20.46772651	19.30164554	5.697169

Table1: Comparison of displacement at mid-point of the beam with formulation of shear stiffness matrix $[K_s]$ computed with 1-Gauss point and 2-Gauss point rule of mesh M2 (tolerance $\varepsilon_D = 10^{-3}$)

The Timoshenko beam theory have a difficulty by using the shear stiffness matrix $[K_s]$ because it may lead to “locking” phenomena with 2-point Gauss-Legendre rule formulation.

4. PLANE STRAIN AND AXISYMMETRIC PROBLEMS IN SOLID MECHANICS APPLICATIONS

4.1 Plane strain problem

Plane strain components: $\varepsilon = [\varepsilon_x , \varepsilon_y , \gamma_{xy}]$

ε_x, ε_y and γ_{xy} :the strain components.

The stress-strain relationships : $\sigma = D\varepsilon$

$$\sigma = [\sigma_x , \sigma_y , \tau_{xy}]^T \quad (24)$$

The stress normal to the xy plane is nonzero

$$\sigma_z = \nu(\sigma_x + \sigma_y) \quad (25)$$

4.2 Axisymmetric problem

The nonzero strains :

$$\varepsilon = [\varepsilon_r , \varepsilon_\theta , \varepsilon_z , \gamma_{rz}]^T \quad (26)$$

For small displacement, the normal strains are given as:

$\varepsilon_r = \dfrac{\partial u}{\partial r}$, $\varepsilon_\theta = \dfrac{u}{r}$, $\varepsilon_z = \dfrac{\partial w}{\partial z}$ and the shear strain is $\gamma_{rz} = \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial r}$

The stress-strain relationships are given as

$$\sigma = D\varepsilon \quad (27)$$

where the stresses $\sigma = [\sigma_r , \sigma_\theta , \sigma_z , \tau_{rz}]^T$ in which σ_r , σ_θ and σ_z are the normal stress in the r, θ and z directions recpectively

τ_{rz} is the shear stress in the rz plane.

4.5 Thick-walled cylinder under internal pressure problem

Problem description:

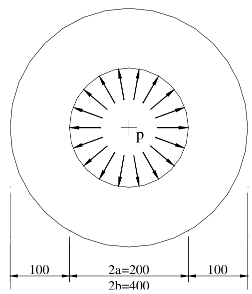


Fig. 6: A thick-walled cylinder under internal pressure

Material properties:

Elastic modulus: $E = 2.1e4\ dN/mm^2$

Poissons ratio : $\nu = 0.3$

Uniaxial yield stress: $\sigma y = 24.0\ dN/mm^2$

Strain hardening parameter : $H' = 0.0$

Geometry proportions:

Internal radius: $a = 100\ mm$

External radius: $b = 200\ mm$

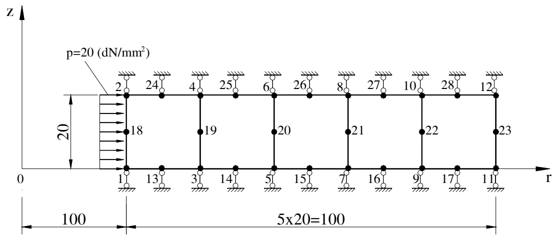


Fig. 7: Finite element idealisation of axisymmetric problem, mesh AM1

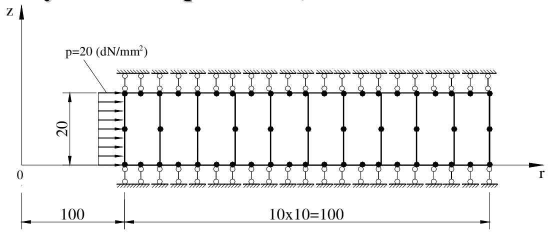


Fig. 8: Finite element idealisation of axisymmetric problem, mesh AM2

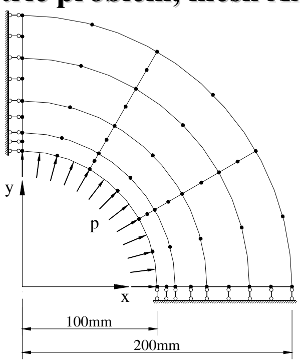


Fig. 9: Finite element idealisation of plane strain problem, mesh PM1

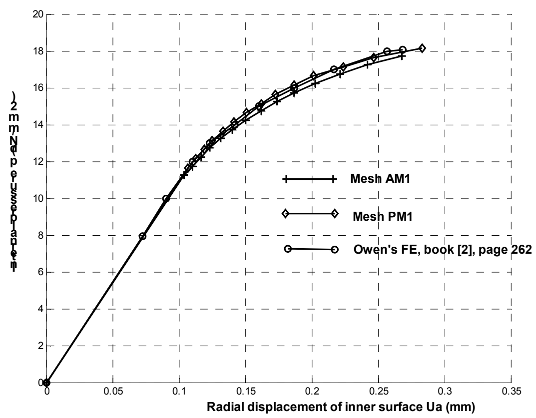


Fig. 10: Radial displacement Ua(mm) of inner face of Mesh AM1, PM1 and Owen's FE

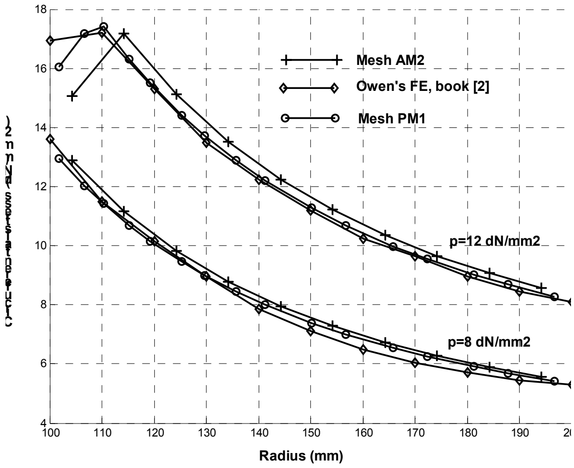


Fig. 11: Comparison of distribution of circumferential stress σ_θ with internal pressure variables p=8 and 12 (dN/mm^2) of mesh AM2, PM1 and Owen's FE

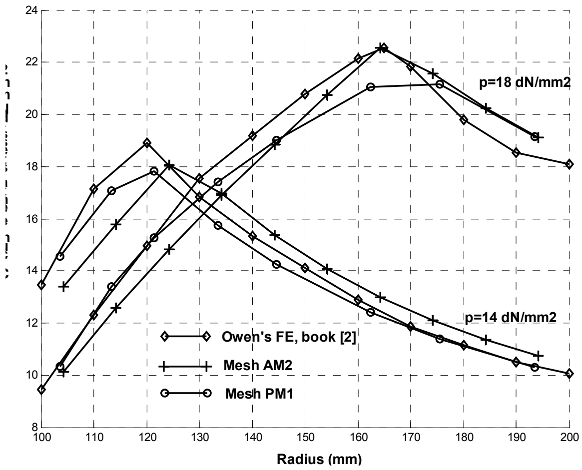


Fig. 12: Comparison of distribution of circumferential stress σ_θ with internal pressure variables p=14 and 18 (dN/mm^2) of mesh AM2, PM1 and Owen's FE

CONCLUSIONS

For the Timoshenko beam problem, the analysis of elasto-plastic behaviour of the beam considered development of plastic zone in beam sections through determining plastic layers. However, the Timoshenko beam theory have a difficulty by using the shear stiffness matrix *[K_S]* because it may lead to "locking" phenomena with 2-point Gauss-Legendre rule formulation. This phenomena can be cured by using 1-point

Gauss-Legendre rule formulation for the shear stiffness matrix. The obtained solutions are sensitive with meshes. The more number of layers the more stiffness of the beam. Unfortunately the solutions obtained by this approach was not checked by experimental results.

For the considered 2-D problem, the results obtained from the present FE of several meshes, even for coarse mesh, is close. However the obtained results of meshes of the axisymmetric problem model is different with the results obtained by the plane strain problem model. The variation stress was rather smooth without concentration of stress.

The modelisation of axisymmetric problem with each element having differential stiffness matrix is especially adaptive for analyzing some thick-walled pipes structures made by composite material. Elements containing differential material properties have differential stiffness, that means they have differential stiffness matrix.

Application of the models can be used to analyse elasto-plastic behaviour for some thick-walled pipes made by composite materials (especially reinforced concrete pipes) and "sandwich" materials.

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