ENERGY-CONSERVING AND DECAYING NEWMARK TEMPORAL INTEGRATION FOR NON-LINEAR DYNAMICS

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ABSTRACT

The traditional Newmark scheme presents deficiencies in the energy conservation and hence a numerical instability might occur when this time-stepping scheme is applied to non-linear transient problems. In order to enhance the stability properties of this implicit time integrator, a simple modification to the Newmark formulas by adding a scalar factor such that errors are compensated in the energy balance.

Recently, toward an efficient temporal integrator, numerical experiments on the performance of the modified scheme is still required. In this paper, the modified Newmark scheme will be extended to dealing with highly non-linear problems such as those involved in large strain or contact/impact problems. The numerical implementation, using a finite element discretization in time and space for hyperelastic material models, verifies the performance of the scheme.

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1. Introduction

Due to its simplicity, the implicit Newmark method is widely used time-stepping algorithm in nonlinear structural analysis. However, the use of the Newmark method might experience a numerical instability when applied to non-linear dynamics problems [4]. Besides the lack of the conservation of the total energy, the Newmark algorithm is also known to loose its numerical dissipation character in the nonlinear regime [1].

In the past, using a strategy, in which the discrete energy balance is enforced, Shanshin Chen et al [7] modified the implicit Newmark scheme to retain unconditional stability in multibody dynamics system analysis. This is contrast to another scheme previously presented Xie et al [4], where the Newmark update is directly embedded in the equation of energy balance.

An alternative strategy to retain unconditional stability in nonlinear region was presented by Bui Q.V et al [5] and denoted as the modified Newmark scheme. It's modification based on the adding a scalar factor to the Newmark scheme, the energy conservation is attained for non-linear problems, hence, the unconditional stability of the scheme is thereby more guaranteed. This factor is obviously dependent upon the combined level of energy, forces and displacement at each time step. Moreover, an another modification for filtering spurious high frequency modes of oscillation in the numerical response but imply the strictly decay of the system total mechanical energy at each time step, meaning that unconditional stability is achieved in the nonlinear regime.

However, this modified Newmark scheme has been successfully applied to N-particles systems as well as one-dimensional bar systems [5]. In the presented study, the modified Newmark scheme will be applied to non-linear large strain or contact/impact problems. Therefore, the algorithmic framework of the scheme will be summarized briefly. Algorithmic modifications will be explained in the context of the isoparametric 2D element.

2. Semidiscrete initial value problem

The initial boundary value problem of non-linear elastodynamics is described by the Cauchy equation of motion, initial values of the displacements and the velocities and boundary

values of Neumann and Dirichlet type. The weak formulation, the spatial discretization by finite elements and the inclusion of viscous damping result in the semidiscrete equation of motion and the corresponding discrete initial values.

$$\mathbf{M}\ddot{\mathbf{d}} + \mathbf{C}\dot{\mathbf{d}} + \mathbf{F}(\mathbf{d}) = \mathbf{R}$$
$$\text{with } \begin{cases} \dot{\mathbf{d}}(t=0) = \dot{\mathbf{d}}_0 \\ \mathbf{d}(t=0) = \mathbf{d}_0 \end{cases} \quad (1)$$

Herein, **M** represents the constant mass matrix, **C,** the constant viscous damping matrix, **F,** the deformation dependent internal forces, **R,** the external forces, **d**, the vector of the structural displacements, $\dot{\mathbf{d}}$, the velocities, $\ddot{\mathbf{d}}$, the accelerations and $\dot{\mathbf{d}}_0$, $\ddot{\mathbf{d}}_0$, initial values, respectively. Consequently, $\mathbf{M}\ddot{\mathbf{d}}$, the inertia forces and $\mathbf{C}\dot{\mathbf{d}}$, the viscous damping forces of the non-linear dynamical system. In this study, the damping matrix is assumed to be a linear combination of the mass **M** and the elastic stiffness $\mathbf{K}^e = \mathbf{K}^m(\mathbf{q} = 0)$ matrices, called proportional or Rayleigh damping (see e.g.[3]), $\mathbf{C} = \alpha_1\mathbf{M} + \alpha_2\mathbf{K}^e$.

3. Time discretization

As the basis for the integration of the semidiscrete initial value problem (1), the time interval of interest [0; *T*] is partitioned in typical time intervals [t_n , t_{n+1}] with the corresponding time step $\Delta t = t_{n+1} - t_n$. Assuming determined state variables $\mathbf{d}_n$; $\dot{\mathbf{d}}_n$; $\ddot{\mathbf{d}}_n$ at the time t_n and prescribed external forces **R**(*t*) for all $t \in [0,T]$ the integration of Eq. (1) is restricted to the successive solution of the state variables at the end of each step $\mathbf{d}_{n+1}$; $\dot{\mathbf{d}}_{n+1}$; $\ddot{\mathbf{d}}_{n+1}$.

3.1 Newmark family

Let β and α be real numbers, $0 \le \beta \le 1/2, 0 \le \gamma \le 1$. The Newmark time stepping scheme is usually written in the following way [3].

$$\begin{aligned} \dot{\mathbf{d}}_{n+1} &= \dot{\mathbf{d}}_n + (1-\gamma)h\ddot{\mathbf{d}}_n + \gamma h\ddot{\mathbf{d}}_{n+1} \\ \mathbf{d}_{n+1} &= \mathbf{d}_n + h\dot{\mathbf{d}}_n + h^2(1/2 - \beta)\ddot{\mathbf{d}}_n \\ &+ h^2\beta\ddot{\mathbf{d}}_{n+1} \end{aligned} \quad (2)$$

Choosing β=0.25 and γ=0.5 leads to average constant acceleration scheme :

$$\begin{aligned} \dot{\mathbf{d}}_{n+1} &= \dot{\mathbf{d}}_n + \frac{h}{2}\left(\ddot{\mathbf{d}}_n + \ddot{\mathbf{d}}_{n+1}\right) \\ \mathbf{d}_{n+1} &= \mathbf{d}_n + h\dot{\mathbf{d}}_n + \frac{h^2}{4}\left(\ddot{\mathbf{d}}_n + \ddot{\mathbf{d}}_{n+1}\right) \end{aligned} \quad (4)$$

This scheme is proved to preserve the total energy for linear structural dynamic system [3]. It also exhibits a second order of accuracy. When a parameter $\alpha>0$ is chosen so that: $\gamma = 1/2 + \alpha$ and $\beta = 1/4(1+\alpha)^2$, the scheme (4) becomes the α-dissipative Newmark algorithm.

$$\begin{aligned} \dot{\mathbf{d}}_{n+1} &= \dot{\mathbf{d}}_n + h\{(1/2 - \alpha)\ddot{\mathbf{d}}_n + (1/2 + \alpha)\ddot{\mathbf{d}}_{n+1}\} \\ \mathbf{d}_{n+1} &= \mathbf{d}_n + h\dot{\mathbf{d}}_n + \frac{h^2}{4}\{2 - (1+\alpha)^2\}\ddot{\mathbf{d}}_n \\ &+ \frac{h^2}{4}(1+\alpha)^2\ddot{\mathbf{d}}_{n+1} \end{aligned} \quad (5)$$

If the parameters are chosen $\alpha \in (0, 1/3)$, the result is proved to feature the numerical dissipation and unconditional stability in the linear regime [3]. However, it might instability in non-linear regime, Crisfield et al. [2] have shown that they do not guarantee the dissipation of energy for all time integration parameters leading to stable time integration in the linear range.

3.2 Implicit Modified conservative Newmark Algorithm

In order to retain the energy balance in the non-linear regime, the scalar number σ that find from enforcing the energy balance through an additional term $\mathbf{F}_{err} = \sigma(\mathbf{q}_{n+1} + \mathbf{q}_n)$ in the integration of forces at each time step.

$$\begin{aligned} \sigma = &\frac{(V_n - V_{n+1})}{\|\mathbf{d}_{n+1}\|^2 - \|\mathbf{d}_n\|^2} - \\ &\frac{[(\mathbf{F}_n + \mathbf{F}_{n+1})]^T(\mathbf{d}_{n+1} - \mathbf{d}_n)}{2\{\|\mathbf{d}_{n+1}\|^2 - \|\mathbf{d}_n\|^2\}} \end{aligned} \quad (6)$$

Herein, V_n , V_{n+1} are the potential energy and $\mathbf{F}_n$, $\mathbf{F}_{n+1}$ are the internal force corresponding to the moments t_{n+1}, t_n. Moreover, the modification is not altered the convergence order of the original Newmark method [5]. The time-stepping scheme (4) becomes the Modified conservative Newmark scheme as follow:

$$\begin{aligned}
&\dot{\mathbf{d}}_{n+1} = \dot{\mathbf{d}}_n + \frac{h}{2}\left(\ddot{\mathbf{d}}_n + \ddot{\mathbf{d}}_{n+1}\right) + \\
&\frac{h\sigma}{2}\mathbf{M}^{-1}\left(\mathbf{d}_n + \mathbf{d}_{n+1}\right) \\
&\mathbf{d}_{n+1} = \mathbf{d}_n + h\dot{\mathbf{d}}_n + \frac{h^2}{4}\left(\ddot{\mathbf{d}}_n + \ddot{\mathbf{d}}_{n+1}\right) \\
&+ \frac{h^2\sigma}{2}\mathbf{M}^{-1}\left(\mathbf{d}_n + \mathbf{d}_{n+1}\right)
\end{aligned} \tag{7}$$

Besides the scalar number σ that is employed for the conservation, a parameter χ>0 is introduced to controllable numerical dissipation, that based on the energy-decaying inequality. As a result, unconditional stability is thereby automatically achieved in the non-linear regime and the time-stepping scheme (5) is modified:

$$\begin{aligned}
&\dot{\mathbf{d}}_{n+1} = \dot{\mathbf{d}}_n + \left(\frac{h}{2} - \frac{h\chi}{2}\right)\ddot{\mathbf{d}}_n + \\
&\left(\frac{h}{2} + \frac{h\chi}{2}\right)\ddot{\mathbf{d}}_{n+1} + \frac{h\sigma\mathbf{M}^{-1}}{2}\left(\mathbf{d}_n + \mathbf{d}_{n+1}\right) \\
&\mathbf{d}_{n+1} = \mathbf{d}_n + h\dot{\mathbf{d}}_n + \left(\frac{h^2}{4} - \frac{h^2\chi^2}{4}\right)\ddot{\mathbf{d}}_n + \\
&\left(\frac{h}{2} + \frac{h\chi}{2}\right)^2 \ddot{\mathbf{d}}_{n+1} + \\
&\frac{(1+\chi)h^2\sigma\mathbf{M}^{-1}}{4}\left(\mathbf{d}_n + \mathbf{d}_{n+1}\right)
\end{aligned} \tag{8}$$

4. Spatial discretization by finite element formulation

The 4-node isoparametric quadrilateral element with linear displacement variation of the displacement field within the element.

4.1 Shape functions and nodal variable

The element shape functions are expressed in terms of the element coordinates *(ξ_1,ξ_2)*. The spatial coordinates are expressed in terms of the shape functions and nodal coordinates by

$$\begin{Bmatrix} x(\xi,t) \\ y(\xi,t) \end{Bmatrix} = N_I(\xi)\begin{Bmatrix} x_I(t) \\ y_I(t) \end{Bmatrix} \tag{9}$$

herein $\xi=\{\xi_1,\xi_2\}$. Rate-of-deformation and the velocity approximation is

$$\begin{aligned}
&\begin{Bmatrix} D_{xx} \\ D_{yy} \\ \mathbf{2}D_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial v_x}{\partial x} \\ \dfrac{\partial v_y}{\partial y} \\ \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} \end{Bmatrix} = \\
&\begin{bmatrix} \dfrac{\partial N_I}{\partial x} & 0 \\ 0 & \dfrac{\partial N_I}{\partial y} \\ \dfrac{\partial N_I}{\partial x} & \dfrac{\partial N_I}{\partial y} \end{bmatrix}\begin{Bmatrix} v_{xI} \\ v_{yI} \end{Bmatrix}
\end{aligned} \tag{10}$$

4.2 The internal nodal forces

$$(\mathbf{f}_I^{\text{int}})^T = \begin{bmatrix} f_{xI} & f_{yI} \end{bmatrix}^{\text{int}} = \int_{\Omega} \mathbf{B}_I^T \boldsymbol{\sigma} d\Omega \tag{11}$$

The gradients of the shape functions with respect to the spatial coordinates can then be computed by

$$\begin{aligned}
&\mathbf{B}_I = N_{I,\mathbf{x}}^T = N_{I,\xi}^T \mathbf{x}_{,\xi}^{-1} = \\
&\left[\partial N_I / \partial \xi_i\right]\frac{\mathbf{1}}{J_\xi}\begin{bmatrix} y_{,\xi_2} & -x_{,\xi_2} \\ -y_{,\xi_1} & x_{,\xi_1} \end{bmatrix}
\end{aligned} \tag{12}$$

The integration is performed over the parent domain. For this purpose, we use $d\Omega = J_\xi a d\xi_1 d\xi_2$, where *a* is the thickness, so

$$\begin{aligned}
&\left(\mathbf{f}_I^{\text{int}}\right)^T = \begin{bmatrix} f_{xI} & f_{yI} \end{bmatrix}^{\text{int}} = \\
&\int_{\Xi} \begin{bmatrix} N_{I,x} & N_{I,y} \end{bmatrix}\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} a J_\xi d\Xi
\end{aligned} \tag{13}$$

4.3 Mass matrix

$$\widetilde{M} = \int_{\Omega_0} [N_I]^T [N_I] \rho_0 d\Omega_0 \qquad (14)$$

We use $d\Omega_0 = J_\xi^0(\xi_1, \xi_2) a_0 d\xi_1 d\xi_2$, where $J_\xi^0(\xi_1, \xi_2)$ is the determinant of the Jacobian of the transformation of the parent element to the initial configuration and a_0 is the thickness of the undeformed element. A lumped, diagonal mass matrix can be obtained by apportioning the total mass of the element equally among the 4 nodes.

5. Numerical validation

5.1 Problem descriptions

The problem is defined in Fig. 1. It is a bar in plan stress state , this bar impacts a rigid wall with an initial velocity v_0 in vertical direction. Due to a zero gap between the contact bodies, the only non-linearity of the problem finds its origin from the material model. The chosen material parameters, the geometry and all reference values are given in Table 1.

Fig. 1 also shown the discretization of the bar. We distinguish two domains of the discretization , namely the upper domain ABCD at the impact zone , where 3x10 finite elements are located , and lower domain CDEF , where 3x7 elements discretize the bar.

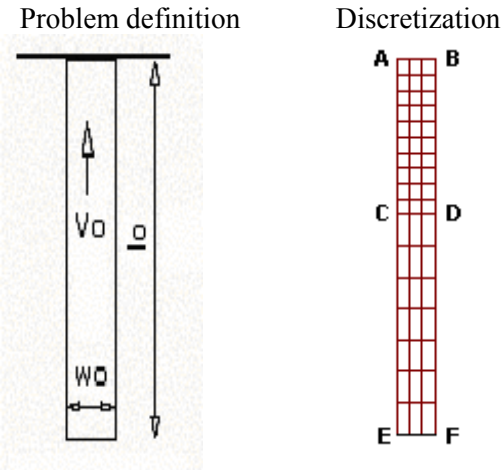
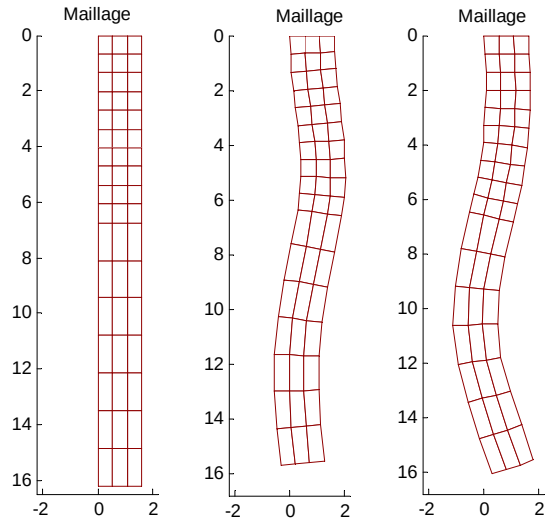


Fig. 1: Problem definition and discretization.

Let us consider St. Venant-Kirchhoff material, that are considered to be compressible with **E** denotes the material strain tensor and **C** is fourth-order elasticity strain energy density function given by

$$w(\mathbf{C}) = \frac{1}{2}\mathbf{E} : \mathbf{C} : \mathbf{E} \qquad (15)$$

Table 1 Material parameters and Geometry value
Young's modulus : E = 2100N/cm^2
Poisson's ratio : ν = 0.3
Mass density : ρ = 8930 kg/m^3
L_0 = 16.2 cm ; w_0 = 1.6 cm
T_0 = 0.1cm ; v_0 = 0.227 m/s



$t = 0$ $t = 0.85$ $t = 1.0$

Fig. 2: Some stages of the bar's deformation .

The four-node isoparametric elements are employed to solve the problem. Fig. 2 shows a sequence of deformed configurations corresponding to differences of instant t = 0, 0.85 and 1s.

5.2 Remarks

(1) It can be observed the large deformation of the impact bar. The Fig. 3 shows the evolution of the energy of the bar during impact. Clearly, with the use of many size of time step h = 0.005 and h = 0.02, many oscillations of energy are observed in the solution predicted by the Newmark method. In contrast, the energy is constant with the modified Newmark update. The Fig. 4 shows the changing between the kinetic and internal energy under enforcing

conservative total energy when the modified Newmark scheme is employed. In that case, without presence of external forces, the energies of the system are under controlled.

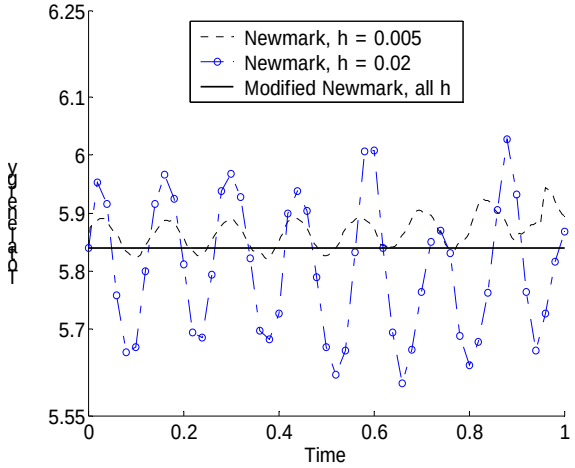


Fig. 3: Energy evolution of the impact bar with the Saint Venant-Kirchhoff material.

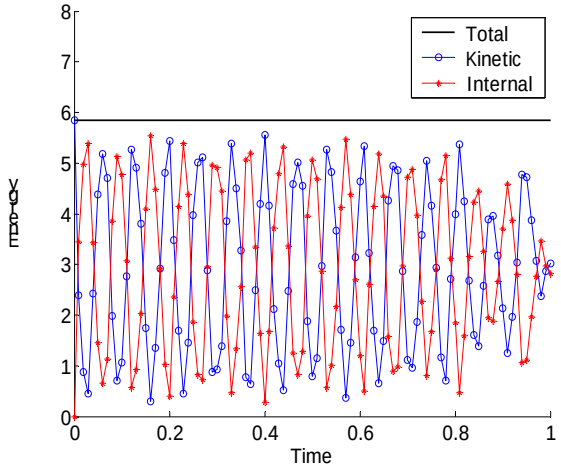


Fig. 4: Evolution of energies of the impact bar using the modified Newmark scheme

(2) Let us investigate the accuracy of the time-stepping schemes: to reduce the effort of calculation but do not alter the nature of the problem, we calculate the relative displacement error (%) at the end of interval $T = 0.5$s.

The displacement error of the bar's tip (point A) $\mathbf{e}_d(t) = \mathbf{d}_{app}(t) - \mathbf{d}_{ref}(t)$ between the approximate solution $\mathbf{d}_{app}$ and reference solution $\mathbf{d}_{ref}$ is recorded. The reference solution is obtained with sufficiently small time steps of size $h = 0.0001$, where results from different time-stepping schemes are practical identical.

The results are governed in table 2. The results show the accuracy of the Newmark and modified Newmark are seem to be the same.

Table 2
Displacement error of the tip A

Time step	Modified Newmark	Origin Newmark
	Node 4-x	Node 4-x
5.10^{-4}	1.55993	1.55994
10^{-3}	1.56289	1.56292
$2.5.10^{-3}$	1.57396	1.57404
5.10^{-3}	1.59180	1.59192

Table 3
Error of displacement (%)of the tip A.

Time step	Ref. solution	Modified NMK	Origin NMK
5.10^{-4}	1.55985	0.0001	0.0001
10^{-3}	1.55985	0.0020	0.0019
$2.5.10^{-3}$	1.55985	0.0091	0.0090
5.10^{-3}	1.55985	0.0206	0.0205

(3) Due to the presence of high-frequency oscillations, in case of using timestep $h = 0.005$, if a damping $\chi = 0.005$ is introduced into the system, the total energy is expected to decay. In agreement with this argument, the modified dissipative Newmark scheme predicts a steady decay of energy.

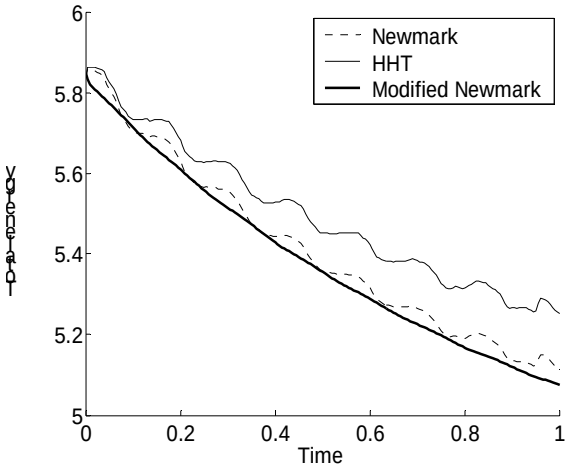


Fig. 5 : Decay of energy due to numerical damping ($\chi = 0.005$)

In contrast, the traditional α-dissipative Newmark and HHT schemes exhibit oscillating energy curves with clear rises at the beginning and the end (Fig. 5).

(4) When using time step $h = 0.0004$, the response of velocity component of the tips at both ends in the horizontal and vertical direction, that given by the Newmark time-stepping scheme is very noisy (Fig. 7).

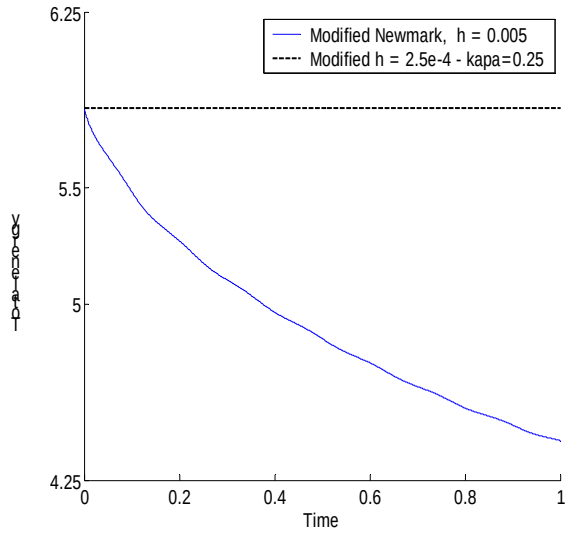


Fig. 6: Decay of energy due to numerical damping ($\chi = 0.25$)

Let us consider alternative approaches of smoothing transient responses without resorting to numerical dissipation ($\chi = 0.0$).

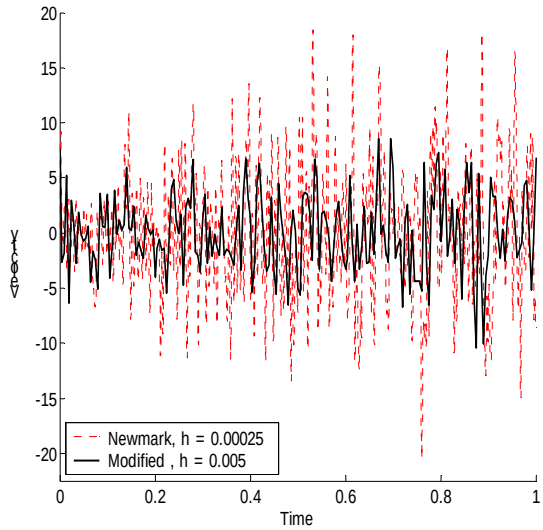


Fig. 7: Speed response of a tip A in x-axis from Newmark scheme is very noisy

The use of a large step size $h = 0.005$ allows to smooth out the high frequency modes, yet the energy conserving feature of the solution remains unaffected (Fig.7-9). This smoothing resulted from so-called non-dissipative damping.

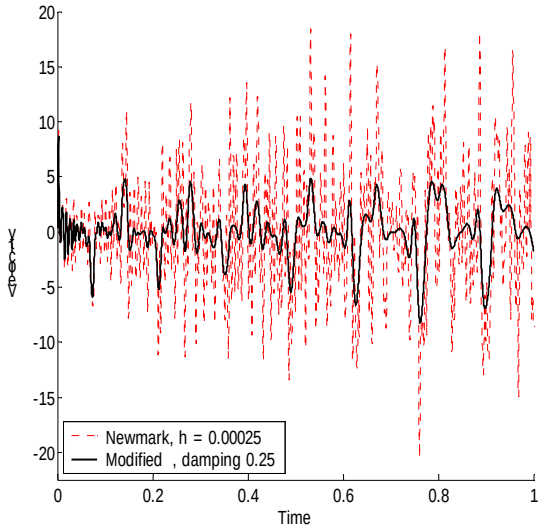


Fig. 8: Horizontal speed responses of tip A of the impact bar ($h=0.0004$).

Zienkiewicz et al. [6] assumed that an accurate time integration of forces enables not to significantly excite the high-frequency modes and this procedure a form of damping in the modes if large a time step is used.

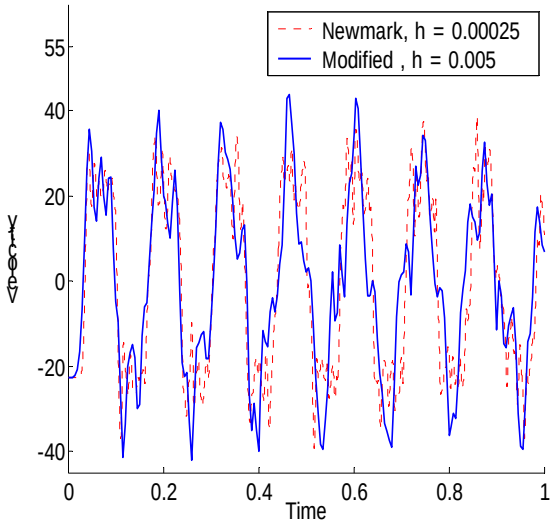


Fig 9: Smoothed–out speed response of a tip E in the y-axis with $\chi = 0$.

For modified Newmark scheme, since the internal forces are better integrated, the resulting time-stepping scheme might hence possess the mentioned damping effect as far as the algorithm can still maintain key qualitative features of low-frequencies responses. However,

because low and high modes are tightly coupled in our case so that affection is limited.

Although the responses are still noisy, however, we must have use an important amount of numerical damping ($\chi = 0.25$) to smooth out these responses (Fig. 8-10), hence, the benefit of numerical damping in smoothing out dynamic responses is obvious , hence, a correct decay of the total energy is occurred (Fig. 6).

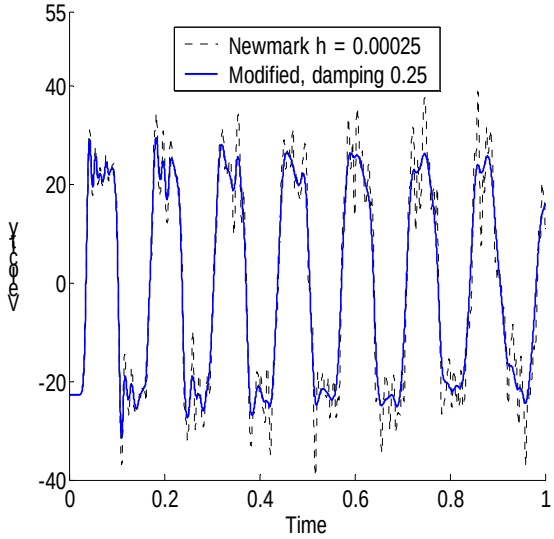


Fig 10: Smoothed–out the speed response tip E with numerical damping($\chi = 0.25$)

6. Conclusions

The application of the modified Newmark to the nonlinear elastodynamics using the 2D isoparametric quadrilateral element with non-linear hyperelastic material models was presented.

Nevertheless, toward an efficient temporal integrator and before any attempt can be made for a larger system, the performance of the modified scheme for 3D solid and friction/frictionless contact problems and improve the reduction of accuracy when numerical damping is introduced in the modified Newmark algorithm [5] need be examined in the future.

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